

# Option Pricing Model for Incomplete Market

**Sergei Fedotov**

Department of Mathematics  
UMIST PO Box 88  
Manchester M60 1QD  
United Kingdom

**Sergei Mihkailov**

Bergische Universität  
FB-14, Gausstr. 20, 42097 Wuppertal  
Deutschland

## ABSTRACT

The problem of determining the European-style option price in the incomplete market has been examined within the framework of stochastic optimization. An analytic method based on the discrete dynamic programming equation (Bellman equation) has been developed that gives the general formalism for determining the option price and the optimal trading strategy (optimal control policy) that reduces total risk inherent in writing the option. The basic purpose of the paper is to present an effective algorithm that can be used in practice.

**Keywords:** option pricing, incomplete market, transaction costs, stochastic optimization, Bellman equation

# 1 Introduction

An essential feature of the currently dominant option pricing theory proposed by Black and Scholes is the existence of a dynamic trading strategy in the underlying asset that exactly replicates the derivative contract payoff [1-4]. However, in general, the market is not complete, the contingent claim is not redundant asset and therefore its price cannot be determined by the no-arbitrage argument alone. The reasons that give rise to an incompleteness of market might be very different, for example, mixed jump-diffusion price process for an asset [1,5], stochastic volatility [6], etc.

In recent years there has been a substantial theoretical efforts to give the pricing formula for a derivative security for which an exact replicating portfolio in the underlying asset ceases to exist. The typical example involving incompleteness is a model in which the stock volatility is a stochastic process. Several approaches to the valuation of the contingent claim under random volatility have been suggested in literature [6-11]. Typically the pricing formulas involve the unknown and what is more unobservable parameter, so-called market price of volatility risk. This fundamental difficulty has led the researches to accept the idea of uncertain volatility when all prices for contingent claim are possible within some specified range [12-15].

An alternative method for the derivative pricing in the incomplete market has been proposed in a series of papers by mathematicians Müller, Fölmer, Sondermann, Schweizer and Schäl [16-20] and by physicists Bouchaud and Sornette [21] (see also [22-25]). The basic idea is that the fair price of contingent claim can be found through the risk minimization procedure. Different criteria for a measuring the risk inherent in writing an option have been suggested including the global and local variance of the cost process [16-18] and the variance of the global operator wealth [19,21-23]. We refer to the recent survey paper [20] for an exposition of the status of research on the incomplete market involving the stochastic volatility and risk-minimization.

Although significant progress has already been made in the option pricing theory involv-

ing the risk minimization procedure, still there exist many open problems including how to derive an *effective algorithm* giving the option price and trading strategy. The purpose of this paper is to present such an algorithm that can be used in practice. The aim is to show how the problem of option pricing based on the risk-minimization analysis can be reformulated in terms of Maier problem and how the stochastic optimization procedure [26] based on the Bellman equation can be implemented to give a reliable numerical technique for determining both the derivative price and optimal trading strategy. The application of dynamic programming approach to option pricing can be found in [18,28,29].

## 2 Statement of the problem

We assume a discrete-time  $N$ -period world in which the dynamics of the security price  $S_n$  is governed by the stochastic difference equation

$$S_{n+1} = S_n + \xi_n S_n, \quad n = 0, 1, \dots, N-1, \quad (1)$$

where  $\xi_n$  is a sequence of random variables whose conditional jump density at time  $n$  is independent of the asset price  $S_m$ ,  $m < n$ , and is given by

$$\rho_n(\xi, S_n) \equiv \frac{\partial}{\partial \xi} P\{\xi_n < \xi | S_n\}. \quad (2)$$

We assume that at time 0 an investor sells an European-style option with the strike price  $X$  for  $C_0$  and invests this money in a portfolio containing  $\Delta_0$  shares held long partially financed by borrowing  $B_0$  in cash. The current value of this portfolio is given by

$$V_0 = C_0 = \Delta_0 S_0 - B_0. \quad (3)$$

The investor is interested in constructing the self-financing strategy to hedge the option exposure. Since for the incomplete market the exact replication of the option payouts by a portfolio of traded securities is not possible, the investor cannot completely neutralize the risk inherent in writing the option. Hence the problem is to find such a trading strategy that reduces total risk to some intrinsic value[16-25].

To proceed further we need an equation governing the dynamics of the self-financing hedged portfolio. First we consider the case of frictionless trading (the effect of transaction costs will be also examined in this paper). The value of the portfolio  $V_n$  at time  $n$  may be written as

$$V_n = \Delta_n S_n - B_n, \quad n = 0, 1, \dots, N-1, \quad (4)$$

where  $\Delta_n$  is the number of shares of the underlying asset held long during the time interval  $[n, n+1)$  and  $B_n$  is the amount of money borrowed. At the beginning of trading period  $n+1$  just before readjusting the position this portfolio is worth

$$V_{n+1} = \Delta_n S_{n+1} - (1+r) B_n, \quad n = 0, 1, \dots, N-1, \quad (5)$$

where  $r$  is the interest rate. Therefore the change in the value of the portfolio can be written as

$$V_{n+1} = (1+r) V_n + \Delta_n (\xi_n - r) S_n, \quad n = 0, 1, \dots, N-1. \quad (6)$$

Following [19] we propose that the investor's purpose is to maintain a self-financing portfolio (4) in a such way that at the expiration date  $N$  the terminal value of this portfolio

$$V_N = (1+r) V_{N-1} + \Delta_{N-1} (\xi_{N-1} - r) S_{N-1} \quad (7)$$

should be as close as possible to the option payoff

$$\theta_X(S_N) \equiv \max(S_N - X, 0). \quad (8)$$

One way to achieve this purpose is to require that the expectation value of the difference between the option value and the value of hedged portfolio at expiration is equal to zero, i.e.  $\mathbf{E}\{\theta_X(S_N) - V_N\} = 0$  while the variance of this difference  $\mathbf{E}\{(\theta_X(S_N) - V_N)^2\}$  as a measure for risk should be minimized by the proper choice of the trading strategy; here  $\mathbf{E}\{\cdot\}$  denotes expectation with respect to the distributions of  $\xi_0, \xi_1, \dots, \xi_{N-1}$ .

### 3 Stochastic optimization

According the ideas of dynamic programming [26,27], the proper choice of the sequence controls  $\Delta_n, B_n$  should involve the information aggregation, i.e. the optimal choice of trading strategy at each of  $N$  time period should be based on the available information about the current values of asset price and hedged portfolio. From a mathematical point of view it means that one have to find a sequence of functions (so-called optimal control policy)

$$\Delta_n^* = \Delta_n^*(S_n, V_n), \quad B_n^* = B_n^*(S_n, V_n) \quad n = 0, 1, \dots, N-1 \quad (9)$$

that minimize the total risk. In what follows we will use (4) to find an optimal value of  $B_n$ , that is.

$$B_n^*(S_n, V_n) = \Delta_n^*(S_n, V_n) S_n - V_n, \quad n = 0, 1, \dots, N-1. \quad (10)$$

Instead of the problem of minimizing the risk subject to the constraint we consider the problem of minimizing the modified risk-function

$$R_\lambda \equiv \mathbf{E} \left\{ (\theta_X(S_N) - V_N)^2 + \lambda (\theta_X(S_N) - V_N) \right\}, \quad (11)$$

where  $\lambda$  is the Lagrange multiplier and the average is made again over all  $\xi_n$ .

Following the dynamic programming approach, we consider first the last time period and proceed backward in time. If at the beginning of the last trading period  $N-1$  the stock price is  $S_{N-1}$  and the value of portfolio is  $V_{N-1}$ , then no matter what happened in the past periods, the investor should choose such a trading strategy  $\Delta_{N-1}, B_{N-1}$  that minimizes the risk for the last time period.

Let us introduce the minimal risk for the last period which is a function of the stock price  $S_{N-1}$  and the value of the portfolio  $V_{N-1}$

$$I_0(S_{N-1}, V_{N-1}) = \min_{\Delta_{N-1}} \mathbf{E}_{\xi_{N-1}} \left\{ (\theta_X(S_N) - V_N)^2 + \lambda (\theta_X(S_N) - V_N) \right\}. \quad (12)$$

It follows from (1) and (6) that  $I_0$  can be rewritten as

$$I_0(S_{N-1}, V_{N-1}) =$$

$$\begin{aligned} \min_{\Delta_{N-1}} \mathbf{E}_{\xi_{N-1}} & (\theta_X (S_{N-1} + \xi_{N-1} S_{N-1}) - (1+r) V_{N-1} - \Delta_{N-1} (\xi_{N-1} - r) S_{N-1})^2 \\ & + \lambda (\theta_X (S_{N-1} + \xi_{N-1} S_{N-1}) - (1+r) V_{N-1} - \Delta_{N-1} (\xi_{N-1} - r) S_{N-1}). \end{aligned} \quad (13)$$

By calculating this function we obtain the optimal value of  $\Delta_{N-1}$  and thereby the optimal trading policy  $\Delta_{N-1}^*(S_{N-1}, V_{N-1}), B_{N-1}^*(S_{N-1}, V_{N-1})$  for the last period.

At the beginning of time period  $N-2$  when the stock price is  $S_{N-2}$  and the value of portfolio is  $V_{N-2}$  the investor should readjust the position in a such way that  $(\Delta_{N-2}, B_{N-2})$  minimize the risk  $\mathbf{E}_{\xi_{N-2}} \{I_0(S_{N-1}, V_{N-1})\}$ .

The dynamic programming algorithm takes the form of the recurrence relation

$$\begin{aligned} I_1(S_{N-2}, V_{N-2}) = \\ \min_{\Delta_{N-2}} \mathbf{E}_{\xi_{N-2}} \{I_0(S_{N-2} + \xi_{N-2} S_{N-2}, (1+r) V_{N-2} + \Delta_{N-2} (\xi_{N-2} - r) S_{N-2})\}. \end{aligned} \quad (14)$$

By calculating  $I_1(S_{N-2}, V_{N-2})$  we obtain the optimal function  $\Delta_{N-2}^* = \Delta_{N-2}^*(S_{N-2}, V_{N-2})$ .

Repeating these arguments we can get the Bellman equation for the period  $n$

$$I_{N-n}(S_n, V_n) = \min_{\Delta_n} \mathbf{E}_{\xi_n} \{I_{N-n-1}(S_n + \xi_n S_n, (1+r) V_n + \Delta_n (\xi_n - r) S_n)\}. \quad (15)$$

The last equation can be rewritten in the form

$$I_{N-n}(S_n, V_n) = \min_{\Delta_n} \int I_{N-n-1}(S_n + \xi S_n, (1+r) V_n + \Delta_n (\xi - r) S_n) \rho_n(\xi, S_n) d\xi. \quad (16)$$

The attractive feature of the dynamic programming algorithm is the relative simplicity with which the optimal trading policy  $\Delta_n^* = \Delta_n^*(S_n, V_n), B_n^* = B_n^*(S_n, V_n), n = 0, 1, \dots, N-1$  can be computed. The basic advantage of general algorithm (15) over functional derivative technique [21] is that the original problem (11) is reduced to a sequence of minimization problems which of them is much simpler than the original one.

It might seem that the better choice of control in (15) would be a pair  $(\Delta_n, B_n)$  giving the control policy  $\Delta_n^*(S_n), B_n^*(S_n)$  as the functions of the asset price  $S_n$  only. However the self-financing condition gives rise to the restriction

$$\Delta_n S_n - B_n = \Delta_{n-1} S_n - (1+r) B_{n-1} \quad (17)$$

which makes the control problem in terms of the pair  $(\Delta_n, B_n)$  rather difficult.

It is clear that the function  $I_N(S_0, V_0)$  is the minimal risk for the optimal trading strategy when the initial value of stock is  $S_0$  and the value of portfolio is  $V_0$ . The initial investment required to fund the partially hedged portfolio is nothing else but the price of option  $C_0$  which can be determined from the equation

$$\frac{\partial I_N(S_0, C_0)}{\partial C_0} = 0. \quad (18)$$

The discussion of when the optimal initial investment  $C_0$  can be considered as a fair option price and problems that might arise from that can be found in [19].

## 4 Transactions costs

Let us now consider the problem of finding the optimal trading strategy and the option price in the presence of transactions costs. We know that the effects of transactions costs on the contingent claim pricing might be very complex depending on the size of bid-offer spreads, the structure of payoff functions, etc.[4,30-32]. Here we suggest a new algorithm for a valuation of option price based on the risk minimization procedure.

We assume a bid-offer spread in which the investor buys the stock for the offer price  $S(1+k)$  and sells it for the bid price  $S(1-k)$ . Again we formulate the problem in terms of an investor who sells the European option with payout  $\theta_X(S_N)$  and who employs the trading strategy to hedge the derivative. At time 0 a hedged portfolio is constructed by purchasing of  $\Delta_0$  shares at the offer price  $S_0(1+k)$  and borrowing  $B_0$  in cash at the riskless rate  $1+r$ , so that the ammount of money spent for this portfolio including the effect of transaction cost can be written as

$$V_0 = \Delta_0 S_0 - B_0 + k \Delta_0 S_0. \quad (19)$$

It is assumed here that the investor has no initial position in the underlying asset. The investor's purpose is to maintain a dynamic portfolio strategy in a such way that the risk of his liability (11) is minimal.

Above we have derived a Bellman equation (15) when the asset price and the value of portfolio have been chosen as the dynamical variables while the number of shares in portfolio has played the role of the control parameter. In the presence of transaction costs it is more convenient to make another choice of basic variables and controls. It follows from the self-financing condition that the stochastic dynamics of the amount of dollars  $B_n$  borrowed can be written as

$$B_{n+1} = (1 + r) B_n + (\Delta_{n+1} - \Delta_n + k | \Delta_{n+1} - \Delta_n |) S_{n+1}, \quad n = 0, 1, \dots, N - 1. \quad (20)$$

Let us introduce a new control parameter  $\Omega_n$  such that

$$\Delta_{n+1} = \Omega_n, \quad n = 0, 1, \dots, N - 1, \quad (21)$$

then the dynamical variables  $B_n$  obeys the stochastic difference equation

$$B_{n+1} = (1 + r) B_n + (\Omega_n - \Delta_n + k | \Omega_n - \Delta_n |) (1 + \xi_n) S_n, \quad n = 0, 1, \dots, N - 1. \quad (22)$$

Now we are in a position to formulate the basic problem. Let us denote by  $I_N(S_0, B_0, \Delta_0)$  the minimal risk that can be achieved by starting from the arbitrary initial state  $S_0, B_0, \Delta_0$

$$I_N(S_0, B_0, \Delta_0) = \min_{\Omega_0, \dots, \Omega_{N-1}} \mathbf{E} \left\{ (\theta_X(S_N) - \Delta_N S_N + B_N)^2 + \lambda (\theta_X(S_N) - \Delta_N S_N + B_N) \right\}. \quad (22)$$

Then the principle of optimality yields the general recurrence relation

$$I_{N-n}(S_n, B_n, \Delta_n) = \min_{\Omega_n} \mathbf{E}_{\xi_n} \{ I_{N-n-1}(S_n + \xi_n S_n, (1 + r) B_n + (\Omega_n - \Delta_n + k | \Omega_n - \Delta_n |) (1 + \xi_n) S_n, \Omega_n) \}. \quad (23)$$

Clearly, the optimal values of  $\Delta_0$  and  $B_0$  determining the initial investment  $V_0$  as a fair option price can be determined by

$$\frac{\partial I_N(S_0, B_0, \Delta_0)}{\partial B_0} = 0, \quad \frac{\partial I_N(S_0, B_0, \Delta_0)}{\partial \Delta_0} = 0. \quad (24)$$



## 5 Summary

To conclude, we have formulated the European-style pricing model for the incomplete market in which the risk incurred by selling an option cannot be completely hedged by dynamic trading. New effective algorithm based on the discrete dynamic programming equation has been presented that gives the option price and optimal trading strategy. The method accommodates the effects of transaction costs and can easily be extended to price options when volatility is random. The uncertain volatility case can be also treated by the stochastic optimization procedure. There are several directions to explore by the method presented here. First, one may study the case with imperfect state information regarding the asset prices. Also, one can study various adaptive control problems. It should be noted that the preliminary work done here might be of big practical importance and therefore merits further investigation including the computational aspect of our formalism.

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